Jan 15, 2023

Week 2

We knew how to evaluate $\iint_R f$ where R is a rectangle.

We'd like to define

$$\iint_D f$$

where D is a region.

A <u>region</u> | <u>domain</u> is the set bounded by one or several curves. It consists of interior points and boundary points.

examples of regions.

Given a function on a set $E \subset \mathbb{R}^2$, we extend it to be a function in $\mathbb{R}^2$ by

$$\tilde{f}(x,y) = \begin{cases} f(x,y) & , (x,y) \in E \\ 0 & , (x,y) \notin E. \end{cases}$$

Given a function f defined on a region D. We pick a rectangle R containing D and define

$$\iint_D f = \iint_R \tilde{f}.$$

We point out

- If f is piecewise continuous on D, $\tilde{f}$ is piecewise continuous in $\mathbb{R}^2$, so $\tilde{f}$ is integrable in any R.

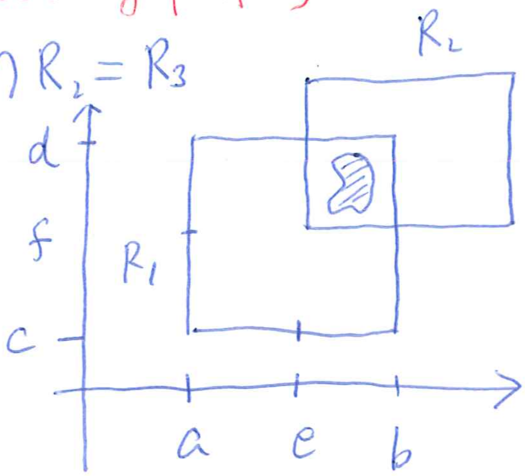
- If $D \subset R_1$, $D \subset R_2$,

$$\iint_{R_1} \tilde{f} = \iint_{R_2} \tilde{f},$$

hence the above definition is indept of the choice of R containing D. (You could skip the following proof.)

Pf: Let $R_1 = [a,b]\times[c,d]$, $R_1 \cap R_2 = R_3$

$R_3 = [e,b]\times[f,d]$.



Consider a partition P on R_1 taking e and f as endpoints. Then all subrectangles of P inside R_3 form a partition of R_3. Denote it by P'.

Consider the Riemann sum

$$S(\tilde{f},P) = \sum_{P} \tilde{f}(P_{ij})\Delta x_i \Delta y_j$$

$$= \sum_{P'} \tilde{f}(P_{ij})\Delta x_i \Delta y_j + \sum_{\substack{\text{outside}\\ P'}} \tilde{f}(P_{ij})\Delta x_i \Delta y_j .$$

Pick P_{ij} to be the center of R_{ij}, then $\tilde{f}(P_{ij}) = 0$ for R_{ij} not inside P'. Therefore,

$$S(\tilde{f},P) = \sum_{P'} \tilde{f}(P_{ij})\Delta x_i \Delta y_j$$

$$= S(\tilde{f},P')$$

As $\|P\| \to 0$, LHS $\to \iint_{R_1} f$, RHS $\to \iint_{R_3} f$

We conclude $\iint_{R_1} \tilde{f} = \iint_{R_3} \tilde{f}$.

Similarly, can show

$$\iint_{R_2} \tilde{f} = \iint_{R_3} \tilde{f}$$

$\therefore \iint_{R_1} \tilde{f} = \iint_{R_2} \tilde{f}$. ▨

When D is described as

$$\{(x,y) : g_1(x) \leq y \leq g_2(x),\ a \leq x \leq b\},$$

then

$$\iint_D f = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\, dy\, dx.$$

▨ Pf: $\iint_D f \overset{\text{def}}{=} \iint_R \tilde{f}$, $D \subset R = [a,b] \times [c,d]$.

$$= \int_a^b \left(\int_c^d \tilde{f}(x,y)\, dy \right) dx \quad \text{(Fubini's thm)}$$

$$= \int_a^b \left(\int_c^{g_1(x)} \tilde{f}(x,y)\, dy + \int_{g_1(x)}^{g_2(x)} \tilde{f}(x,y)\, dy + \int_{g_2(x)}^{d} \tilde{f}(x,y)\, dy \right) dx$$

$$= 0 + \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\, dy\, dx + 0,$$

since $\tilde{f}(x,y) = 0$ when $c \leq y \leq g_1(x)$, $\tilde{f}(x,y) = 0$ when $g_2(x) \leq y \leq d$, and $\tilde{f}(x,y) = f(x,y)$ when $g_1(x) \leq y \leq g_2(x)$, $\forall x \in [a,b]$.

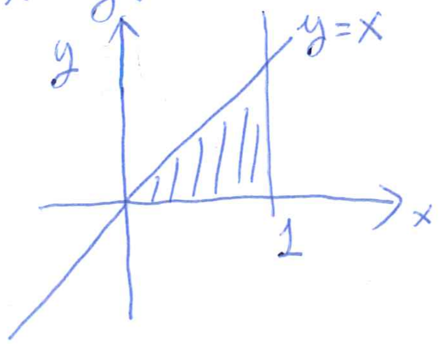
▨

Switching x and y, when

$$D = \{(x,y) : h_1(y) \le x \le h_2(y),\ c \le y \le d\},$$

$$\iint_D f = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy.$$

e.g. Find the volume of the prism whose base is the triangle in xy-plane bdd by $y=x$, $x=1$, and the x-axis, and whose top is $z = 3-x-y$.



the triangle can be described as

$$T = \{(x,y) : 0 \le y \le x,\ 0 \le x \le 1\}$$

$$\therefore \text{volume} = \iint_T (3-x-y)\,dA(x,y)$$

$$= \int_0^1 \int_0^x (3-x-y)\,dy\,dx = \int_0^1 \left(3y - xy - \frac{y^2}{2}\right)\Big|_0^x dx$$

$$= \int_0^1 \left(3x - \frac{3}{2}x^2\right) dx = \left(3\frac{x^2}{2} - \frac{3}{2}\frac{x^3}{3}\right)\Big|_0^1 = 1.$$

Alternatively,

$$T = \{(x,y) : y \le x \le 1,\ 0 \le y \le 1\}$$

$$\therefore \text{volume} = \iint_T (3-x-y)\,dA(x,y)$$

$$= \int_0^1 \int_y^1 (3-x-y)\,dx\,dy = \int_0^1 \left(\frac{5}{2} - 4y + \frac{3}{2}y^2\right) dy$$

$$= \left(\frac{5}{2}y - 2y^2 + \frac{3}{2}\frac{y^3}{3}\right)\Big|_0^1 = 1.$$

e.g. Evaluate

$$\iint_T \frac{\sin x}{x}\, dA$$

where T is the triangle in the previous example.

$$\iint_T \frac{\sin x}{x}\, dA = \int_0^1 \int_0^x \frac{\sin x}{x}\, dy\, dx$$

$$= \int_0^1 \frac{\sin x}{x}\, x\, dx = -\cos x\Big|_0^1$$

$$= -\cos 1 + 1 \approx 0.46.$$

On the other hand,

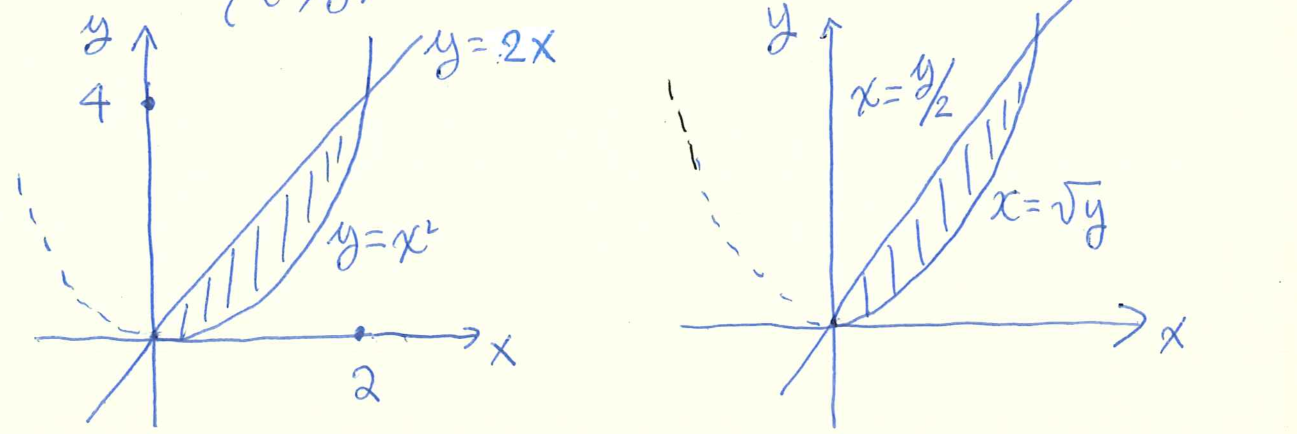
$$\iint_T \frac{\sin x}{x}\, dA = \int_0^1 \int_y^1 \frac{\sin x}{x}\, dx\, dy$$

can't be done. So the order of integration matters!

e.g. $\int_0^2 \int_{x^2}^{2x} (4x+2)\, dy\, dx$. Change the order of integration.

The region is $\{(x,y) : x^2 \le y \le 2x,\ 0 \le x \le 2\}$

ie.



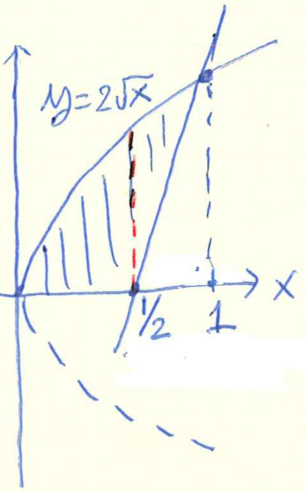
The region can be described as

$$\{(x,y) : \sqrt{y} \le x \le y/2 \ , \ 0 \le y \le 4\}.$$

Hence

$$\int_0^2 \int_{x^2}^{2x} (4x+2)\,dy\,dx = \int_0^4 \int_{y/2}^{\sqrt{y}} (4x+2)\,dx\,dy .$$

e.g. Find the volume of the solid lying beneath $z = 16 - x^2 - y^2$ and above the region D bounded by $y = 2\sqrt{x}$, $y = 4x - 2$, and the x-axis.



D can be decomposed into

$$D_1 = \{(x,y) : 0 \le y \le 2\sqrt{x}, \ 0 \le x \le \tfrac{1}{2}\}$$

and

$$D_2 = \{(x,y) : 4x - 2 \le y \le 2\sqrt{x}, \ \tfrac{1}{2} \le x \le 1\}$$

Hence

$$\iint_D (16 - x^2 - y^2)\,dA = \iint_{D_1} (16 - x^2 - y^2)\,dA + \iint_{D_2} (16 - x^2 - y^2)\,dA$$

$$= \int_0^{\frac{1}{2}} \int_0^{2\sqrt{x}} (16 - x^2 - y^2)\,dy\,dx + \int_{\frac{1}{2}}^{1} \int_{4x-2}^{2\sqrt{x}} (16 - x^2 - y^2)\,dy\,dx$$

$$\vdots$$

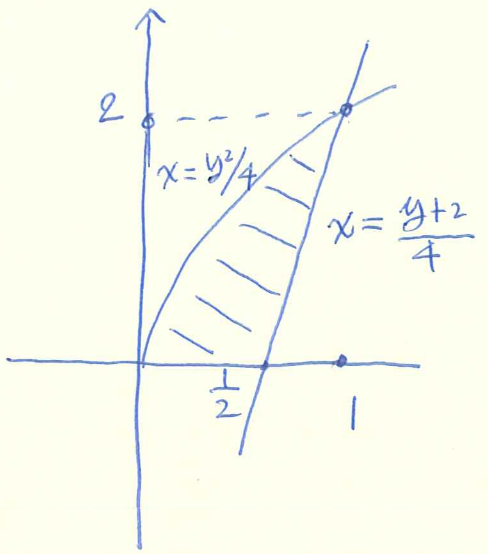
$$= \frac{20803}{1680} \approx 12.4.$$

On the other hand, D can be described as

$$D = \{(x,y) : y^2/4 \le x \le \frac{y+2}{4} \ , \ 0 \le y \le 2\}$$

$$\therefore \iint_D (16-x^2-y^2)\,dA = \int_0^2 \int_{y^2/4}^{y+2/4} (16-x^2-y^2)\,dx\,dy$$

$$= \frac{20803}{1680}.$$



Three basic properties of integrals.

(linearity) Ⓐ $\iint_D (\alpha f + \beta g)\,dA = \alpha \iint_D f + \beta \iint_D g\,dA, \quad \forall \alpha, \beta \in \mathbb{R}.$

(positivity) Ⓑ $f \geq 0$ on $D \Rightarrow \iint_D f\,dA \geq 0$

(additivity) Ⓒ Let a curve C divide D into 2 regions D_1 and D_2. Then for continuous f,

$$\iint_D f\,dA = \iint_{D_1} f\,dA + \iint_{D_2} f\,dA$$

An application of the double integral is it enables us to define the area of a region rigorously.

For a region D, we define its area to be

$$\text{area} = \iint_D 1 \, dA$$

$$\left(= \iint_R \tilde{1} \, dA \right).$$

So, the concept of an area is well-defined whenever $\tilde{1}$ is integrable.

To justify this definition, we let $D \subset R$ and P a partition on the rectgle R.

Let

$\mathcal{A}$ = R_{ij} lying inside D

$\mathcal{B}$ = R_{ij} touching both D and outside of D

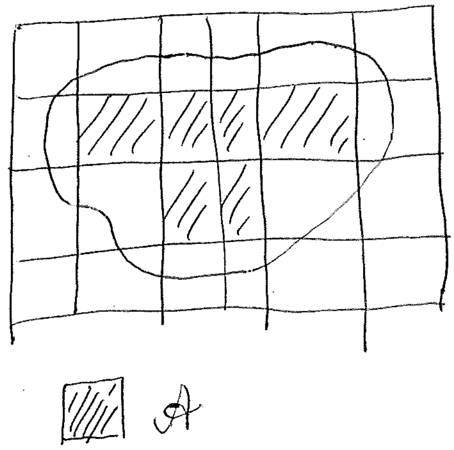
$\mathcal{C}$ = R_{ij} lying outside D.

Define "outer approximate area" of D

$$\sum_{\mathcal{A}} \Delta x_i \Delta y_j + \sum_{\mathcal{B}} \Delta x_i \Delta y_j ,$$

and "inner approximate area" of D,

$$\sum_{\mathcal{A}} \Delta x_i \Delta y_j .$$



<u>Theorem</u> As $\|P\| \to 0$, the outer and inner approximate areas approach $\iint_D 1\, dA$ (provided $\tilde{1}$ is integrable).

▨ Pf. It suffices to show both approx. areas are Riemann sums of $\tilde{1}$.

Indeed, if pick $P_{ij} \in D$ when $R_{ij} \in \mathcal{A}$ or $\mathcal{B}$,

$$R(\tilde{1}, P) = \sum_{\mathcal{A}} \tilde{1}(P_{ij})\, \Delta x_i \Delta y_j + \sum_{\mathcal{B}} \tilde{1}(P_{ij})\, \Delta x_i \Delta y_j + \underbrace{\sum_{\mathcal{C}} \tilde{1}(P_{ij})\, \Delta x_i \Delta y_j}_{\text{always } = 0}$$

$$= \sum_{\mathcal{A}} \Delta x_i \Delta y_j + \sum_{\mathcal{B}} \Delta x_i \Delta y_j$$

$$= \text{outer approx. area},$$

if pick $P_{ij} \notin D$ when $R_{ij} \in \mathcal{B}$, $\tilde{1}(P_{ij}) = 0$, so

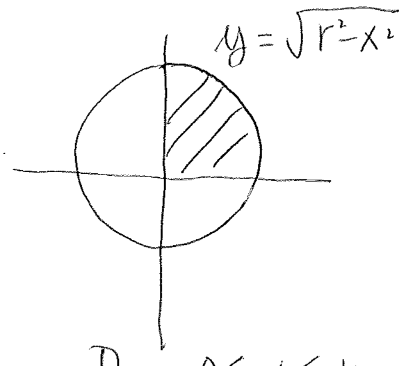
$$R(\tilde{1}, P) = \sum_{\mathcal{A}} \Delta x_i \Delta y_j + \sum_{\mathcal{B}} 0\, \Delta x_i \Delta y_j$$

$$= \sum_{\mathcal{A}} \Delta x_i \Delta y_j$$

$$= \text{inner approx. area.} \quad ▨$$

We show this def. is consistent with old formula,

e.g. The area of a disk of radius r is πr^2.



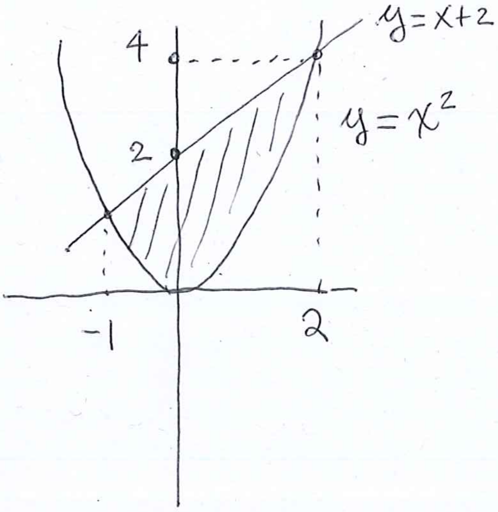
$$\text{area} = \iint_{\text{disk}} 1\, dA = 4 \iint_D 1\, dA$$

D : $0 \le x \le r$
$0 \le y \le \sqrt{r^2 - x^2}$

$$= 4 \int_0^r \int_0^{\sqrt{r^2 - x^2}} dy\, dx = 4 \int_0^r \sqrt{r^2 - x^2}\, dx$$

$$= 4 \times \int_0^{\pi/2} r^2 \cos^2\theta\, d\theta = \pi r^2. \#$$

e.g. Find the area of the region enclosed by $y = x^2$, and $y = x + 2$.



$$\text{area} = \iint_D dA$$

$$= \int_{-1}^{2} \int_{x^2}^{x+2} dy\, dx$$

$$= \int_{-1}^{2} (x + 2 - x^2)\, dx$$

$$\vdots$$

$$= \frac{1}{6}.$$

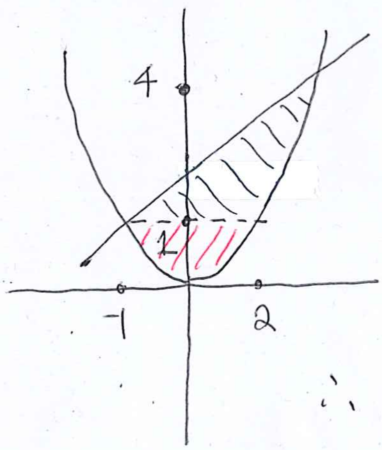
Or, consider the following decomposition: $D = D_1 \cup D_2$

D_1 : $0 \le y \le 1$,

$-\sqrt{y} \le x \le \sqrt{y}$

D_2 : $1 \le y \le 4$

$y - 2 \le x \le \sqrt{y}$



$$\therefore \text{area} = \iint_{D_1} dA + \iint_{D_2} dA$$

$$= \int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} dx\, dy + \int_1^4 \int_{y-2}^{\sqrt{y}} dx\, dy$$

$$\vdots$$

$$= \frac{1}{6}.$$

Obviously, the first approach is simpler.

eg. Find the area of the playing field

$$-2 \leq x \leq 2$$

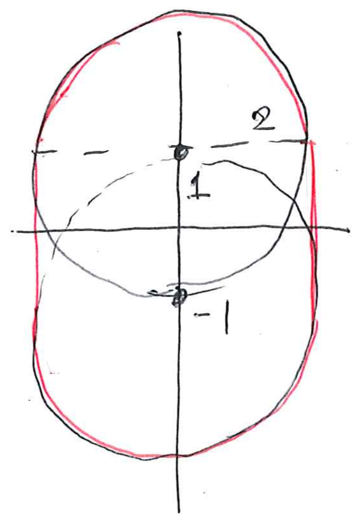
$$-1-\sqrt{4-x^2} \leq y \leq 1+\sqrt{4-x^2}.$$

Here $g_1(x) = -1-\sqrt{4-x^2}$, $g_2(x) = 1+\sqrt{4-x^2}$

$y = -1-\sqrt{4-x^2}$, describes the lower side of $(y+1)^2+x^2=4$

$y = 1+\sqrt{4-x^2}$ describe the upper side of $(y-1)^2+x^2=4$.

$$\text{area} = \int_{-2}^{2}\int_{-1-\sqrt{4-x^2}}^{1+\sqrt{4-x^2}} 1\, dy\, dx$$



$$= 4\int_0^2\int_0^{1+\sqrt{4-x^2}} dy\, dx$$

$\vdots$

$$= 8+4\pi.$$